About minimal fractions

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Abstract

Starting with concrete problems related to fractions with minimal denominator, laying in given interval, in this note we will consider their generalizations with correspondent theory and solutions of it in algorithmic spirit.

I. Introduction.
As introduction we will start from two concrete problems with consideration of different ways to solve them.
Problem1.
Let m, n be positive integers such that $\frac{7}{10} < \frac{m}{n} < \frac{11}{15}$.
Find the smallest possible value of n.
Solution 1.
$\frac{7}{10} < \frac{m}{n} < \frac{11}{15} \iff \begin{cases} 10m - 7n = k \\ 11n - 15m = l \\ k, l \in \mathbb{N} \end{cases} \implies$
$3(10m - 7n) + 2(11n - 15m) = 3k + 2l \iff$
$n = 3k + 2l$ and $11(10m - 7n) + 7(11n - 15m) = 11k + 7l \iff$
$5m = 11k + 7l$.
Since $5(m - (2k + l)) = k + 2l$ then minimal value of $3k + 2l$, where $k, l \in \mathbb{N}$ and $k + 2l$ is divisible by 5 can be attained only if $k = 1, l = 2$. That is $\min n = 7$ and correspondent $m = 5$ and, therefore, desired fraction is $\frac{5}{7}$.
Solution 2.
Consider interval (α, β) where $0 < \alpha$ and two transformation.
Transformation 1 (in the case $\beta < 1$.)

$$T_1: \quad If I = (\alpha, \beta) \text{ then } T_1(I) = \left(\frac{1}{\beta}, \frac{1}{\alpha}\right) \text{ and if } \alpha < \frac{p}{q} < \beta \text{ then} T_1\left(\frac{p}{q}\right) = \frac{q}{p};$$

Transformation 2.(in the case $\alpha > 1$).

$$T_2: \text{ If } I = (\alpha, \beta) \text{ then } T_2(I) = (\alpha - [\alpha], \beta - [\alpha]) \text{ and } if\ \alpha < \frac{p}{q} < \beta \text{ then} T_2\left(\frac{p}{q}\right) = \frac{p}{q} - [\alpha].$$

Applying both transformation to $\left(\frac{7}{10}, \frac{11}{15}\right)$ we obtain
$T_1\left(\frac{7}{10}, \frac{11}{15}\right) = \left(\frac{15}{11}, \frac{10}{7}\right), T_2\left(\frac{15}{11}, \frac{10}{7}\right) = \left(\frac{15}{11} - 1, \frac{10}{7} - 1\right) = \left(\frac{4}{11}, \frac{3}{7}\right)$,
$T_1\left(\frac{4}{11}, \frac{3}{7}\right) = \left(\frac{7}{3}, \frac{11}{4}\right), T_2\left(\frac{7}{3}, \frac{11}{4}\right) = \left(\frac{7}{3} - 2, \frac{11}{4} - 2\right) = \left(\frac{1}{3}, \frac{3}{4}\right)$,
$T_1\left(\frac{1}{3}, \frac{3}{4}\right) = \left(\frac{4}{3}, 3\right), T_2\left(\frac{4}{3}, 3\right) = \left(\frac{4}{3} - 1, 3 - 1\right) = \left(\frac{1}{3}, 2\right)$.
Fraction with minimal denominator in $\left(\frac{1}{3}, 2\right)$ is $\frac{1}{1}$.
Then, applying in reverse inverse transformations to 1 we obtain:
$1 \longmapsto 2 \longmapsto \frac{1}{2} \longmapsto \frac{1}{2} + 2 = \frac{5}{2} \longmapsto \frac{2}{5} \longmapsto \frac{2}{5} + 1 = \frac{7}{5} \longmapsto \frac{5}{7}$.
We can see that $\frac{7}{10} < \frac{5}{7} \iff 49 < 50$ and $\frac{5}{7} < \frac{11}{15} \iff 75 < 77$.
We will prove that no fractions between $\frac{7}{10}$ and $\frac{11}{15}$ with denominator $n < 7 \iff n \le 6$.
Really, assuming that such fraction exist we obtain
$\frac{7}{10} < \frac{m}{n} \iff 7n < 10m \iff \left[\frac{7n}{10}\right] + 1 \le m$.
From the other hand $\frac{m}{n} < \frac{11}{15} \iff m \le \left[\frac{11n}{15}\right]$.
Thus $\left[\frac{7n}{10}\right] + 1 \le \left[\frac{11n}{15}\right]$ but for $n = 6, 5, 4$ this inequality becomes, respectively,
$\left[\frac{7 \cdot 6}{10}\right] + 1 \le \left[\frac{11 \cdot 6}{15}\right] \iff 5 \le 4, \left[\frac{7 \cdot 5}{10}\right] + 1 \le \left[\frac{11 \cdot 5}{15}\right] \iff 4 \le 3$,
$\left[\frac{7 \cdot 4}{10}\right] + 1 \le \left[\frac{11 \cdot 4}{15}\right] \iff 3 \le 2$.
Also obvious that $\frac{2}{3} < \frac{7}{10}$.Therefore, minimal denominator is 7.

Solution 3.
$\frac{7}{10} < \frac{m}{n} < \frac{11}{15} \implies \frac{15}{11} < \frac{n}{m} < \frac{10}{7} \iff \frac{4}{11} < \frac{n-m}{m} < \frac{3}{7} \implies$
$\frac{7}{3} < \frac{m}{n-m} < \frac{11}{4} \implies \frac{1}{3} < \frac{m}{n-m} - 2 < \frac{3}{4} \iff$
$\frac{1}{3} < \frac{3m-2n}{n-m} < \frac{3}{4} \implies \frac{4}{3} < \frac{n-m}{3m-2n} < 3 \implies$
$\frac{1}{3} < \frac{n-m}{3m-2n} - 1 < 2 \iff \frac{1}{3} < \frac{3n-4m}{3m-2n} < 2$.
Since fraction with minimal denominator in $\left(\frac{1}{3}, 2\right)$ is $\frac{1}{1}$ then
from claim $\begin{cases} 3n - 4m = 1 \\ 3m - 2n = 1 \end{cases}$ we obtain $n = 7, m = 5$ and further, as in **Solution 2.**

Problem2.
Rational number represented by irreducible fraction $\frac{p}{q}$ belong to interval $\left(\frac{6}{13}, \frac{7}{15}\right)$.Prove that $q \geq 28$.
This problem has the following interpretation:
Prove that $\min\left\{q \mid q \in \mathbb{N} \text{ and } \exists\,(p \in \mathbb{N})\left[\frac{6}{13} < \frac{p}{q} < \frac{7}{15}\right]\right\} = 28$.
We start from two following statement related to positive integers a, b, c, d and represented here in the form of problems, offered to the reader as an exercise
1. Prove that for any fraction $\frac{a}{b}, \frac{c}{d}$ such that $\frac{a}{b} < \frac{c}{d}$ holds inequality
$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.
2. Prove that if $bc - ad = 1$ then $\frac{a}{b}, \frac{c}{d}$ both irreducible and
$\frac{a+c}{b+d}$ is irreducible as well. (In our problem $7 \cdot 13 - 15 \cdot 6 = 1$).
We generalize original problem in the form of the following

Theorem.
Let $\frac{a}{b}$ and $\frac{c}{d}$ be two positive fraction such that $\frac{a}{b} < \frac{c}{d}$ and $bc - ad = 1$ and irreducible fraction $\frac{p}{q}$ belong to interval $\left(\frac{a}{b}, \frac{c}{d}\right)$.
Then $q \geq b + d$.
Proof.
First note that $c(b+d) - d(a+c) = b(a+c) - a(b+d) = bc - ad = 1$.
Assume that there is a fraction $\frac{p}{q}$ such that $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$ and $q < b + d$.
Since $\frac{a}{b} < \frac{p}{q} \implies pb - aq > 0 \iff pb - aq \geq 1$ and
$\frac{p}{q} < \frac{c}{d} \implies qc - pd > 0 \iff qc - pd \geq 1$
then $d(pb - aq) + b(qc - pd) \geq b + d \iff$
$q(bc - ad) \geq b + d \iff q \geq b + d$.
Thus, we obtain the contradiction $q \leq b + d < b + d$ which complete the proof.■
That is $\min\left\{q \mid q \in \mathbb{N} \text{ and } \exists\,(p \in \mathbb{N})\left[\frac{a}{b} < \frac{p}{q} < \frac{c}{d}\right]\right\} = b + d$.
We can prove even more, namely prove that fraction
$\frac{p}{q} \in \left(\frac{a}{b}, \frac{c}{d}\right)$ with smallest denominator $q = b + d$ defined uniquely and $p = a + c$.
Also we can see that for any fraction $\frac{p}{q}$ such that $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$
and $bc - ad = 1$ holds $q \geq b + d$ and
$p \geq a + c$ $(c(pb - aq) + a(qc - pd) \geq a + c \iff$
$p(bc - ad) \geq a + c \iff p \geq a + c)$.

Let $\dfrac{p}{b+d}$ be fraction with minimal denominator $b+d$ such that
$\dfrac{a}{b}<\dfrac{p}{b+d}<\dfrac{c}{d}$.
Assume that $p>a+c$. Since $0<c(b+d)-pd \iff 1\leq c(b+d)-pd$
then $1\leq c(b+d)-pd<c(b+d)-(a+c)d=bc-ad=1$,
that is contradiction.
Therefore, $p=a+c$ and fraction with minimal denominator
defined uniquely and equal to $\dfrac{a+c}{b+d}$.

Remark.
In the case $0<\dfrac{a}{b},\dfrac{c}{d}$ such that $\dfrac{a}{b}<\dfrac{c}{d}$ and $bc-ad\neq 1$ the way
of finding of "internal" fraction with minimal denominator, represented
above isn't works.

II. Problem in general. Theory and algorithms.
So, the purpose of these notes is consideration of the ways to solve
of the following general problem
What is the smallest possible denominator for fractions $\dfrac{m}{n}$ *belonging to the open interval* (α,β) *where* $\alpha<\beta$ *are given positive real numbers?*
We introduce the following definitions and notations:
Interval (a,b) will be called *naturally filled (* $\mathbb{N}-$ *filled)* if there is at least one natural number m such that $a<m<b$ $((a,b)\cap\mathbb{N}\neq\varnothing)$.
Easy to see that (a,b) is $\mathbb{N}-$ *filled* if and only if $\lfloor a\rfloor+1<b$.
Indeed, since $\lfloor a\rfloor\leq a<m \implies \lfloor a\rfloor+1\leq m$ and $m<b$ then $\lfloor a\rfloor+1<b$;
In case $\lfloor a\rfloor+1<b$ interval (a,b) is obviously $\mathbb{N}-$ *filled*.
Also we denote via $p(\alpha,\beta)$ the smallest natural n such that interval $(n\alpha,n\beta)$ is $\mathbb{N}-$ *filled*, (that is $p(\alpha,\beta):=\min\{n\mid n\in\mathbb{N}\text{ and }\lfloor n\alpha\rfloor+1<n\beta\}$) and via $q(\alpha,\beta)$ the smallest natural m such that $(p(\alpha,\beta)\alpha<m<p(\alpha,\beta)\beta)$.
Obvious that $q(\alpha,\beta)=\lfloor p(\alpha,\beta)\alpha\rfloor+1$ and then $\alpha<\dfrac{q(\alpha,\beta)}{p(\alpha,\beta)}<\beta$.
From definitions $p(\alpha,\beta)$ and $q(\alpha,\beta)$ immediately follows that fraction $\dfrac{q}{p}$ where $p=p(\alpha,\beta)$ and $q=q(\alpha,\beta)$ is irreducible, because if $d(p,q)=k\neq 1$ then for $n=p/k\in\mathbb{N}$ interval $(n\alpha,n\beta)$ is $\mathbb{N}-$ *filled* and $n<p=p(\alpha,\beta)$ $-$ minimal natural such that $(n\alpha,n\beta)$ is $\mathbb{N}-$ *filled.*
Also, fraction $\dfrac{q}{p}$ is the *minimal fraction* among all fraction which belong to interval (α,β) in the following sense:

For any fraction $\dfrac{m}{n}$ such $\alpha<\dfrac{m}{n}<\beta$ holds inequalities $p\leq n$ and $q\leq m$.

Indeed, since $\alpha<\dfrac{m}{n}<\beta \iff \alpha n<m<\beta n$ then $(\alpha n,\beta n)$ is $\mathbb{N}-$ *filled* and, therefore, by definition $p\leq n$.

Also, since $\alpha p\leq\alpha n<m$ then $\lfloor\alpha p\rfloor<m \implies q=\lfloor\alpha p\rfloor+1\leq m$.

Thus the fraction $\dfrac{q}{p}$ fully justifies the name *minimal fraction.*

*(Geometrically this means that if (α,β) is given interval and $h=\dfrac{1}{n}, n\in\mathbb{N}$ is the step of uniform grid on $\mathbb{R}$, then smallest n, which provide hit at least one node of a grid inside of interval (α,β) ,is $p(\alpha,\beta)$ and $q(\alpha,\beta)$ is a number of node closest to α.)

Now we consider two important properties of minimal fraction.

Property1.

If fraction $\dfrac{q}{p}$ is minimal in (α,β) then fraction $\dfrac{p}{q}$ is minimal in $\left(\dfrac{1}{\beta},\dfrac{1}{\alpha}\right)$.

Proof.

Since $\dfrac{m}{n}$ is minimal in $\left(\dfrac{1}{\beta},\dfrac{1}{\alpha}\right)$ then $\dfrac{n}{m}\in(\alpha,\beta)$ and, therefore, $p\le n,\quad q\le m.$

Since $\dfrac{q}{p}$ is minimal in (α,β) then $\dfrac{p}{q}\in\left(\dfrac{1}{\beta},\dfrac{1}{\alpha}\right)$ and, therefore, $m\le q,\ n\le p$.Hence, $p=n, q=m,$ that is

(1) $\qquad p(\alpha,\beta)=q\left(\dfrac{1}{\beta},\dfrac{1}{\alpha}\right)$ and $q(\alpha,\beta)=p\left(\dfrac{1}{\beta},\dfrac{1}{\alpha}\right)$.

Property 2.

If fraction $\dfrac{q}{p}$ is minimal in (α,β) then for any integer $k\ge-\lfloor\alpha\rfloor$ fraction $\dfrac{q+kp}{p}$ is minimal in $(\alpha+k,\beta+k)$ as well.

Proof.

Let $\dfrac{m}{n}$ is minimal in $(\alpha+k,\beta+k)$ and $\dfrac{q}{p}$ is minimal in (α,β).

Since $\dfrac{m-kn}{n}\in(\alpha,\beta)$ and $\dfrac{q}{p}$ is minimal in (α,β) then, $p\le n$ and $q\le m-kn$.

Since $\dfrac{q}{p}\in(\alpha,\beta)\iff\dfrac{q+kp}{p}\in(\alpha+k,\beta+k)$ then $n\le p$ and $m\le q+kp$.

Thus, $p=n$ and $q+kp\ge m\ge q+kn=q+kp\implies m=q+kp$. So,

(2) $\qquad p(\alpha+k,\beta+k)=p(\alpha,\beta)$ and $q(\alpha+k,\beta+k)=q(\alpha,\beta)+kp(\alpha,\beta)$.

First we consider simple but important case when interval (α,β) is $\mathbb{N}-$ *filled.* Then we obviously have $p(\alpha,\beta)=1$ and $q(\alpha,\beta)=\lfloor\beta\rfloor+1$.

Further we assume that interval (α,β) isn't $\mathbb{N}-$ *filled* ,that is $\lfloor a\rfloor+1\ge b$.

For some such intervals (α,β) values $p(\alpha,\beta)$ and $q(\alpha,\beta)$ also can be obtained in a close form.

Lemma 1 .

Let $\alpha>1$. Then $p(1,\alpha)=\left\lfloor\dfrac{\alpha}{\alpha-1}\right\rfloor, q(1,\alpha)=\left\lfloor\dfrac{\alpha}{\alpha-1}\right\rfloor+1.$

Proof.

Since interval $(n,n\alpha)$ is $\mathbb{N}-$ *filled* iff

$n+1<n\alpha\iff1<n(\alpha-1)\iff\dfrac{1}{\alpha-1}<n\iff$

$\left\lfloor \dfrac{1}{\alpha-1} \right\rfloor + 1 \le n \iff \left\lfloor \dfrac{\alpha}{\alpha-1} \right\rfloor \le n$ then $p(1,\alpha) = \left\lfloor \dfrac{\alpha}{\alpha-1} \right\rfloor$ and
$q(1,\alpha) = \left\lfloor \left\lfloor \dfrac{\alpha}{\alpha-1} \right\rfloor \cdot 1 \right\rfloor + 1 = \left\lfloor \dfrac{\alpha}{\alpha-1} \right\rfloor + 1.$

Corollary 1.
Let n is positive integer and $n < \alpha < n+1$ then for interval (n,α) holds

$$p(n,\alpha) = \left\lfloor \frac{\alpha-n+1}{\alpha-n} \right\rfloor,\ q(n,\alpha) = n\left\lfloor \frac{\alpha-n+1}{\alpha-n} \right\rfloor + 1.$$

Proof.
By **Lemma 1** and **Property 2** we have

$$p(n,\alpha) = p(1,\alpha-n+1) = \left\lfloor \frac{\alpha-n+1}{\alpha-n} \right\rfloor$$

and then $q(n,\alpha) = \lfloor np(n,\alpha) \rfloor + 1 = n\left\lfloor \dfrac{\alpha-n+1}{\alpha-n} \right\rfloor + 1.$

Corollary 2.
Let $0 < \alpha < 1$.Then $p(\alpha,1) = \left\lfloor \dfrac{1}{1-\alpha} \right\rfloor + 1$ and $q(\alpha,1) = \left\lfloor \dfrac{1}{1-\alpha} \right\rfloor$.

Proof.
Applying **Lemma 1** to interval $\left(1, \dfrac{1}{\alpha}\right)$ we obtain

$$p\left(1,\frac{1}{\alpha}\right) = \left\lfloor \frac{1}{1-\alpha} \right\rfloor,\ q\left(1,\frac{1}{\alpha}\right) = \left\lfloor \frac{1}{1-\alpha} \right\rfloor + 1.$$

Then by **Property 1** we obtain $p(\alpha,1) = q\left(1,\dfrac{1}{\alpha}\right) = \left\lfloor \dfrac{1}{1-\alpha} \right\rfloor + 1$ and
$q(\alpha,1) = p\left(1,\dfrac{1}{\alpha}\right) = \left\lfloor \dfrac{1}{1-\alpha} \right\rfloor$.

Corollary 3.
Let (α,β) isn't $\mathbb{N}-$ *filled* interval, such that $\lfloor\alpha\rfloor + 1 = \beta$.Then

$$p(\alpha,\beta) = \left\lfloor \frac{1}{1-\{\alpha\}} \right\rfloor + 1,\ q(\alpha,\beta) = \lfloor\alpha\rfloor + \left\lfloor \frac{1}{1-\{\alpha\}} \right\rfloor (\lfloor\alpha\rfloor + 1)$$

Proof.

1. If α is integer then, $\beta = \alpha + 1$ and, obviously,
$p(\alpha,\beta) = 2,\ q(\alpha,\beta) = 2\alpha + 1.$
If α isn't integer then by **Property 2** and **Corollary 2** we obtain

$$p(\alpha,\beta) = p(\alpha,\ \lfloor\alpha\rfloor + 1) = p(\{\alpha\},1) = \left\lfloor \frac{1}{1-\{\alpha\}} \right\rfloor + 1 \text{ and}$$

$$q(\alpha,\beta) = q(\alpha,\ \lfloor\alpha\rfloor + 1) = q(\{\alpha\},1) + \lfloor\alpha\rfloor p(\alpha,\beta) =$$
$$\left\lfloor \frac{1}{1-\{\alpha\}} \right\rfloor + \lfloor\alpha\rfloor \left(\left\lfloor \frac{1}{1-\{\alpha\}} \right\rfloor + 1 \right) = \lfloor\alpha\rfloor + \left\lfloor \frac{1}{1-\{\alpha\}} \right\rfloor (\lfloor\alpha\rfloor + 1).$$

Note that formulas

$$p(\alpha,\beta) = \left\lfloor \frac{1}{1-\{\alpha\}} \right\rfloor + 1,\ q(\alpha,\beta) = \lfloor\alpha\rfloor + \left\lfloor \frac{1}{1-\{\alpha\}} \right\rfloor (\lfloor\alpha\rfloor + 1)$$

which we obtain in supposition α isn't integer gives right result in case α is integer as well. Thus $p(\alpha,\beta)$ and $q(\alpha,\beta)$ for any positive interval (α,β) such that $\lfloor\alpha\rfloor+1\leq\beta$ or $\lfloor\alpha\rfloor+1>\beta$ and α is integer represented in close form by formulas obtained above.

Lemma 2.
Let $\lfloor\alpha\rfloor+1>\beta$ and α isn't integer. And let $\alpha':=\dfrac{1}{\{\beta\}},\beta':=\dfrac{1}{\{\alpha\}}$.
If $\lfloor\alpha'\rfloor+1>\beta'$ and α' isn't integer then $p(\alpha,\beta)>p\left(\alpha',\beta'\right)$.
Proof.
Since $\lfloor\alpha\rfloor+1>\beta$ yields $\lfloor\alpha\rfloor=\lfloor\beta\rfloor$ then $\alpha=\{\alpha\}+\lfloor\alpha\rfloor,\beta=\{\beta\}+\lfloor\alpha\rfloor$ and, by **Properties 1,2** we obtain

$$p(\alpha,\beta)=p(\{\alpha\},\{\beta\})=q\left(\frac{1}{\{\beta\}},\frac{1}{\{\alpha\}}\right)=q\left(\alpha',\beta'\right)\text{ and}$$
$$q(\alpha,\beta)=q(\{\alpha\},\{\beta\})+\lfloor\alpha\rfloor p(\alpha,\beta).$$

Denoting $\alpha'':=\dfrac{1}{\{\beta'\}},\beta'':=\dfrac{1}{\{\alpha'\}}$
we obtain $p(\alpha,\beta)=q\left(\alpha',\beta'\right)=p\left(\alpha'',\beta''\right)+\lfloor\alpha'\rfloor p\left(\alpha',\beta'\right)$.
Since $\lfloor\alpha'\rfloor\geq1$ and $p\left(\alpha'',\beta''\right)\geq1$ then

$$p(\alpha,\beta)\geq p\left(\alpha',\beta'\right)+1\iff p(\alpha,\beta)>p\left(\alpha',\beta'\right).\blacksquare$$

Let $\alpha_0:=\alpha,\beta_0:=\beta$ and suppose that we already have two sequences $\alpha_0,\alpha_1,...,\alpha_n,\ \beta_0,\beta_1,...,\beta_n$ such that $\lfloor\alpha_k\rfloor+1>\beta_k$ and α_k isn't integer, $k=0,1,2,...,n$ where
$\alpha_{k+1}=\dfrac{1}{\{\beta_k\}},\beta_{k+1}=\dfrac{1}{\{\alpha_k\}},k=0,1,2,...,n-1.$
Let $\dfrac{q_k}{p_k}$ is minimal fraction in interval (α_k,β_k),that is
$p_k=p(\alpha_k,\beta_k),q_k=q(\alpha_k,\beta_k),k=0,1,...,n.$
Then $\lfloor\alpha_k\rfloor=\lfloor\beta_k\rfloor,k=0,1,2,...,n$ and by **Properties 1,2** we have
$p_k=p(\alpha_k,\beta_k)=p(\{\alpha_k\},\{\beta_k\})=q\left(\alpha_{k+1},\beta_{k+1}\right)=q_{k+1},$
$q_k=q(\alpha_k,\beta_k)=q(\{\alpha_k\},\{\beta_k\})+\lfloor\alpha_k\rfloor p(\alpha_k,\beta_k)=$
$p\left(\alpha_{k+1},\beta_{k+1}\right)+\lfloor\alpha_k\rfloor p(\alpha_k,\beta_k)=p_{k+1}+\lfloor\alpha_k\rfloor p_k,k=0,1,2,...,n-1.$
Since $\lfloor\alpha_n\rfloor+1>\beta_n$ and α_n isn't integer then denoting

$$\alpha_{n+1}:=\frac{1}{\{\beta_n\}},\beta_{n+1}:=\frac{1}{\{\alpha_n\}}$$

we obtain interval $\left(\alpha_{n+1},\beta_{n+1}\right)$ with minimal fraction $\dfrac{q_{n+1}}{p_{n+1}}$, and by **Properties 1,2** we have $p_n=q_{n+1}$ and $q_n=p_{n+1}+\lfloor\alpha_k\rfloor p_n$.
If $\lfloor\alpha_{n+1}\rfloor+1\leq\beta_{n+1}$ or $\lfloor\alpha_{n+1}\rfloor+1>\beta_{n+1}$ and α_{n+1} is integer then we can stop process and calculate
$p_{n+1}=p\left(\alpha_{n+1},\beta_{n+1}\right),q_{n+1}=q\left(\alpha_{n+1},\beta_{n+1}\right)$ by obtained above formulas and by reversing procedure obtain $p(\alpha,\beta)$ and $q(\alpha,\beta)$, or otherwise to continue construction of sequence of intervals (α_n,β_n) with correspondent minimal fractions $\dfrac{q_n}{p_n}$.
But sequence of intervals (α_n,β_n) obtained by such way can't be infinite because otherwise, accordingly to **Lemma 2,** we obtain infinite strictly decreasing sequence

of natural numbers $p_0 > p_1 > ... > p_n > ....$,i.e. contradiction.
Thus, after finite numbers of such steps we obtain for some n sequences $\alpha_0, \alpha_1, ..., \alpha_n$, $\beta_0, \beta_1, ..., \beta_n$ such that
$\lfloor \alpha_k \rfloor + 1 > \beta_k$ and α_k isn't integer, $k = 0, 1, 2, ..., n$ and
$\lfloor \alpha_{n+1} \rfloor + 1 \le \beta_{n+1}$ or $\lfloor \alpha_{n+1} \rfloor + 1 > \beta_{n+1}$ and α_{n+1} is integer.
Then $\dfrac{q_{n+1}}{p_{n+1}}$ is minimal fraction in $(\alpha_{n+1}, \beta_{n+1})$ obtained by formulas above.
Using recurrences $p_{k-1} = q_k, q_{k-1} = p_k + \lfloor \alpha_{k-1} \rfloor q_k$, $k = n+1, n, ..., 1$ and starting from p_{n+1}, q_{n+1} we consequentially obtain $p_n, q_n, ..., p_1, q_1, p_0, q_0$.
We can simplify algorithm by the following way:
Using p_{n+1} , $p_n = q_{n+1}$ and recurrence

$$p_{k-2} = p_{k-1} + \lfloor \alpha_{k-1} \rfloor p_k, k = n+1, n-1, ..., 2$$

we can obtain $p(\alpha, \beta) = p_0$.Then, $q(\alpha, \beta) = \lfloor p_0 \alpha \rfloor + 1$.
But we will represent another, more efficient way of finding $p(\alpha, \beta)$ and $q(\alpha, \beta)$. Namely,denoting
$x := p_0 = p(\alpha, \beta), y := q_0 = q(\alpha, \beta)$ we obtain $y = p_1 + \lfloor \alpha_0 \rfloor q_1, x = q_1$.
Hence, $q_1 = x, p_1 = y - \lfloor \alpha_0 \rfloor x$ and more generally for any
$k = 1, 2, ..., n+1$ we assume that $p_k = a_k x + b_k y, q_k = c_k x + d_k y$
and then, since

$$p_{k-1} = q_k \iff a_{k-1}x + b_{k-1}y = c_k x + d_k y \iff$$
$$(a_{k-1} - c_k)x + (b_{k-1} - d_k)y = 0 \text{ and } q_{k-1} = p_k + \lfloor \alpha_{k-1} \rfloor q_k \iff$$
$$c_{k-1}x + d_{k-1}y = a_k x + b_k y + \lfloor \alpha_{k-1} \rfloor (c_k x + d_k y) \iff$$
$$(c_{k-1} - a_k - \lfloor \alpha_{k-1} \rfloor c_k)x + (d_{k-1} - b_k - \lfloor \alpha_{k-1} \rfloor d_k)y = 0$$

we obtain the following correlation

$$a_{k-1} = c_k, b_{k-1} = d_k, c_{k-1} = a_k + \lfloor \alpha_{k-1} \rfloor c_k, d_{k-1} = b_k + \lfloor \alpha_{k-1} \rfloor d_k, k = 1, 2, ..., n+1.$$

Thus, $p_k = a_k x + b_k y, q_k = a_{k-1}x + b_{k-1}y$ where a_k and b_k satisfy to the same recurrence $r_k = r_{k-2} - \lfloor \alpha_{k-1} \rfloor r_{k-1}$, $k = 2, ..., n+1$
and from $q_1 = x = a_0 x + b_0 y$, $p_1 = a_1 x + b_1 y = y - \lfloor \alpha_0 \rfloor x$
we obtain $a_0 = 1, a_1 = -\lfloor \alpha_0 \rfloor$ and $b_0 = 0, b_1 = 1$.
Since $a_{k+1}b_k - a_k b_{k+1} = (a_{k-1} - \lfloor \alpha_k \rfloor r_k) b_k - a_k (b_{k-1} - \lfloor \alpha_k \rfloor r_k) = -(a_k b_{k-1} - a_{k-1} b_k)$ then $a_{k+1}b_k - a_k b_{k+1} = (-1)^k (a_1 b_0 - a_0 b_1) = (-1)^{k+1}$
and, therefore, from system

$$\begin{cases} a_{n+1}x + b_{n+1}y = p_{n+1} \\ a_n x + b_n y = q_{n+1} \end{cases}$$

we obtain $\dfrac{q(\alpha, \beta)}{p(\alpha, \beta)} = \dfrac{y}{x} = \dfrac{(-1)^{n+1}(b_{n+1}q_{n+1} - b_n p_{n+1})}{(-1)^{n+1}(a_n p_{n+1} - a_{n+1}q_{n+1})} = \dfrac{b_{n+1}q_{n+1} - b_n p_{n+1}}{a_n p_{n+1} - a_{n+1}q_{n+1}}$.

And one more way:
Let $t_k := \dfrac{q_k}{p_k}$ and $t := t_0$. Also let $r_k := \lfloor \alpha_k \rfloor, k = 0, 1, ..., n$.
Since $q_{k+1} = p_k$ and $p_{k+1} = q_k - \lfloor \alpha_k \rfloor p_k = q_k - r_k p_k$ then

$$t_{k+1} = \frac{q_{k+1}}{p_{k+1}} = \frac{p_k}{q_k - r_k p_k} = \frac{1}{t_k - r_k}.$$

Thus, we have sequence (t_k) of minimal fractions defined by $t_0 = t$ and

$t_{k+1} = \dfrac{1}{t_k - r_k}, \ k = 0, 1, ..., n.$

By consideration first several terms $t_1 = \dfrac{1}{t - r_0}, t_2 = \dfrac{1}{t_1 - r_1} = \dfrac{1}{\dfrac{1}{t - r_0} - r_1} =$

$\dfrac{t - r_0}{(1 + r_0 r_1) - r_1 t}, t_3 = \dfrac{1}{t_2 - r_2} = \dfrac{1}{\dfrac{t - r_0}{(1 + r_0 r_1) - r_1 t} - r_2} = \dfrac{(1 + r_0 r_1) - r_1 t}{t(1 + r_1 r_2) - (r_0 + r_2 + r_0 r_1 r_2)}$

we can see that it makes sense to find t_k in form $t_k = \dfrac{a_{k-1} t + b_{k-1}}{a_k t + b_k}$.

Then from

$t_{k+1} = \dfrac{1}{t_k - r_k} \iff \dfrac{a_k t + b_k}{a_{k+1} t + b_{k+1}} = \dfrac{1}{\dfrac{a_{k-1} t + b_{k-1}}{a_k t + b_k} - r_k} = \dfrac{a_k t + b_k}{(a_{k-1} - r_k a_k) t + b_{k-1} - r_k b_k}$

follows $a_{k+1} = a_{k-1} - r_k a_k, b_{k+1} = b_{k-1} - r_k b_k$.

Also, since $t_1 = \dfrac{1}{t - r_0} = \dfrac{a_0 t + b_0}{a_1 t + b_1}$ we have $a_0 = 0, b_0 = 1, a_1 = 1, b_1 = -r_0$.

From equation $\dfrac{a_n t + b_n}{a_{n+1} t + b_{n+1}} = t_{n+1}$ we obtain

$a_n t + b_n = t_{n+1} a_{n+1} t + b_{n+1} t_{n+1} \iff$

$t(a_n - t_{n+1} a_{n+1}) = b_{n+1} t_{n+1} - b_n \iff t = \dfrac{b_{n+1} t_{n+1} - b_n}{a_n - t_{n+1} a_{n+1}}$.

In addition consider representation of minimal fraction in (α, β) *by continued fraction.*

Since $t_{k+1} = \dfrac{1}{t_k - r_k} \iff t_k = r_k + \dfrac{1}{t_{k+1}}$ then $t_0 = r_0 + \dfrac{1}{t_1} = r_0 + \dfrac{1}{r_1 + \dfrac{1}{t_2}} =$

$$r_0 + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cfrac{1}{t_3}}} = r_0 + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cfrac{1}{...r_n + \cfrac{1}{t_{n+1}}}}}.$$

If $h_k(x) = r_k + \dfrac{1}{x}$ then $t_0 = (h_0 \circ h_1 \circ ... \circ h_n)(t_{n+1})$.

Function $f_k(t) = \dfrac{1}{t - r_k}$ is inverse function for $h_k(t)$.

Indeed,$(f_k \circ h_k)(t) = \dfrac{1}{r_k + \dfrac{1}{t} - r_k} = t$ and

$(h_k \circ f_k)(t) = r_k + \dfrac{1}{f_k(t)} = r_k + t - r_k = t.$

And, at last, some problems to solve.

Problem 3.

a) Find minimal denominator of internal fractions in the interval $\left(\dfrac{97}{36}, \dfrac{96}{35}\right)$.

b) Find minimal denominator of internal fractions in the interval $\left(\sqrt{2}, \dfrac{71}{50}\right)$.

c) Find minimal internal fraction in interval $\left(\dfrac{x^2}{(x+1)^2}, \dfrac{x^2+1}{(x+1)^2-1}\right)$ where $x, x+2$ are prime numbers.

Problem 4.

Find minimal internal fraction in interval (α, β) if:

a) $(\alpha, \beta) = \left(\sqrt{28}, \sqrt{35}\right)$;

b) $(\alpha, \beta) = \left(\dfrac{220}{127}, \sqrt{3}\right)$.

Problem 5.

Find the smallest possible value of $x+y$, if x and y are positive integers such that

$\dfrac{n-1}{n} < \dfrac{x}{y} < \dfrac{n}{n+1}$, $n \in \mathbb{N}$.(Generalization of problem **M436, CRUX).**